REPLY TO MARTIN GARDNER ON MATHEMATICAL REALISM

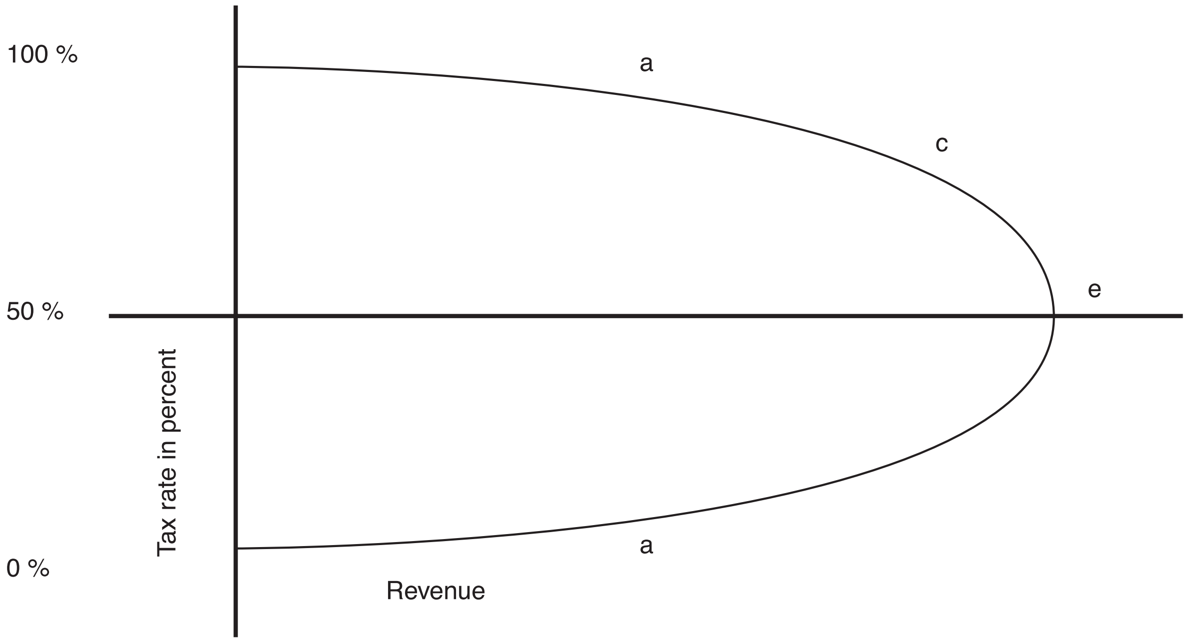

Fig. 1 The Laffer Curve

Why does mathematics describe the world so well? Because that is the way the world really is. We learned mathematics by observing patterns in nature and generalizing them. That, or roughly that, is the position we shall call mathematical realism. A case for it has been eloquently argued by Martin Gardner in many essays and book reviews over many years, some of which have been collected in *The Night Is Large* [St. Martin's Press, New York, 1996, ISBN 0-312-14380-X]. The opposing position Gardner calls collective solipsism (most philosophers call it idealism), and he has taken it to task on many occasions. I shall take mathematical realism to task, but not because I hold the opposing position.

"Because physical laws are based on observations and experiments, they are never absolutely certain. They are always corrigible, subject to possible revision as more is learned about how Nature behaves. Pure mathematics, on the other hand, has a curious kind of certainty. The Pythagorean theorem, for example, is true in all possible worlds because it follows with iron logic from the axioms and rules of a formal system ... It is what philosophers like to call an analytic statement." So writes Gardner in his essay "Mathematics & The Folkways" [*ibid.*] "2+2=4" has a different kind of certainty, not merely a higher degree of certainty, than, for example, "Mars has two moons". "2+2=4" is true independently of anything we might find out about the world (or Mars); "Mars has two moons" is not. But not everyone agrees. Some have argued that mathematics and mathematical truths are no more absolute than *"a code of etiquette, traffic regulations, the rules of baseball, the English language or rules of grammar"* [from Professor White's *The Science of Culture*, as quoted by Gardner]. That is to say, mathematics is wholly a mental construct which we project onto the world, it is not really there in the world, as mathematical realists claim. Let us weigh some of the consequences and presuppositions of these two opposing positions.

One of the arguments in favor of cultural-relativistic mathematics is that different cultures have different mathematical systems. This is a matter of observed fact. For example, not all peoples have counted in base 10—the Babylonians counted in 60s (our reckoning of hours/degrees, minutes, and seconds is a holdover from that), and Gardner himself notes that a certain Indian tribes counted not fingers (tens) but the spaces between fingers (eights). Moreover, the Greek and Roman systems lacked the zero. It is said that there are other tribes where the counting numbers are not infinite, but surprisingly finite: "one, two, three, many" seems to encompass their mathematical horizon. Is this not evidence that mathematics is an arbitrary human construct?

No, Gardner replies, and his argument is cogent. What is different in these different cultures is merely the notation, not the reality. And the reality is that 2+2=4 in any notation, whether in base 10 or base 8 or base 60. The mathematical laws are the same, whatever *numerals* you use. Addition and multiplication are still commutative, for example, and primes are still prime, and whether there is a largest prime—these remain the same, whatever you call them. A rose by any other name would smell as sweet.

So far as I know, we do not know of any culture in which the laws of mathematics are different to ours, not merely the notation. But suppose we came across a tribe in which 2+2 equaled something other than 4 (in whatever notation they used). How would we react to this discovery? With incredulity. Either we would think that they had just gotten it wrong and that if we showed them the correct mathematical procedure, then they would have to agree with it and change their practice, or, if they refused, we would conclude that what they were doing was not mathematics at all, but something else which only superficially resembled mathematics. We would certainly be curious; we would want to know how they treated multiples of two and division by two, and multiples of four and so on, to see whether they made 'consistent errors' or whether they made only the one error whenever they added two twos. We would want to know whether they understood the law of the commutativity of addition and multiplication, for example. We would want to know how wages and rents were paid and whether anybody felt cheated when he bought two pairs of something. In other words, in order for 2+2 to equal something other than 4, many other things would have to be different too, and pretty radically different. So different, in fact, that we would hesitate to call what they were doing "addition" or even "mathematics".

It is different from the case in which other people thought that Mars had a different number of moons than we think it has. If we let them look through our telescopes and they still refused to believe Mars had two moons (they might, for example, say that it was only one moon and its reflection), then we would not be inclined to say that they were not doing astronomy at all, for to deny that Mars has a certain number of moons is not to call any basic laws or assumptions of physical science into question. It is merely to call a single fact into question. We would say that they were certainly doing astronomy, but doing it badly. There is no 'iron logic' that Mars must have a particular number of moons, or any at all for that matter.

It is also different from backgammon or chess; like mathematics, they are formal systems with rules and an 'iron logic'. But different rules of play are known in different countries. Chess in India, for example, was played without castling (until they agreed to change their rules to match the European style). Similarly, some Turks and Iranians play backgammon by slightly different rules than those known to most Europeans and Americans. This presents no problem because no one claims that chess or backgammon is essential to explanatory theories in the physical sciences. We do not feel compelled to call their whole social structure into question, just because they make slightly different moves in chess or backgammon, unlike the case of an imaginary culture which might add 2+2 and get something other than 4, because nothing else of any great consequence hangs on chess or backgammon as it does in the case of 2+2. No one claims that chess is the way it is because the *world* is the way *it* is. Furthermore, the variations are slight; the games are still recognizably chess and backgammon. A few variations of the rules are perfectly acceptable, provided you know what the rules are when you sit down to play. But, we should like to say, such variations of the laws of mathematics would not be acceptable, not even minor ones. Moreover, Mr. Gardner would probably say that such variations are not even possible or thinkable. Why? Because the world isn't like that, not really. Or only because *we* wouldn't call it "mathematics" if it were different?

Gardner poses the following question: if mathematics is solely a human construct projected onto the world, then would the moon still be spherical if no humans existed? Would spiral galaxies still be spiral-shaped if no humans existed? The implications for mathematical solipsists are difficult to avoid. If they say yes, then sphericity and spirality are the thin end of a wedge, the thick end of which appears to commit them to the whole of mathematics, independent of minds. If they say no, then they appear to be committed to George Berkeley's idealism: which is to say, nothing at all exists, not shapes, not quantities, not colors or temperatures, no properties of any kind whatsoever, independent of minds. As Gardner rightly points out, *"if we refuse to say that the form is out there in space, independent of you and me, do we have any right to say that the nebula is out there?"* Berkeley's idealism is logically consistent and, empirically, it changes nothing in the world—every observable fact remains just as it is—but the price is high, for it commits you not only to a peculiar metaphysic, but to a particular theology as well. It commits you to the existence of a God who is not a deceiver, otherwise the universe would disappear if everyone were to blink at the same time. It is a price few sane men are prepared to pay. Nonetheless, I think that Gardner's dilemma is not as exclusive as he supposes; a path might be found between the horns.

The 'iron logic' of mathematics is a well-suited phrase. Iron rusts and bends; a mathematical realist would not like to admit that mathematics sometimes does, too. Of course, I am not suggesting that 2+2 sometimes equals a bit more or a bit less than 4. At the level of simple arithmetic, the laws of mathematics are a good deal harder than iron. But not all of mathematics is as straightforward as that. In higher dimensional topology, for example, which is rather like untying knots in space, more complicated shapes have to be computationally reduced to less complicated shapes, and the procedure for doing this, although it is rigorous, is somewhat arbitrary. The number of variables in a very complicated higher-dimensional shape is prohibitively large, and in order to render it tractable to analysis, a number of them are simply left out of consideration on the assumption that they are inessential. Perhaps a realist would counter that the shape is what it is and the fact that finite minds have to simplify it to understand it shows that the finite minds, but not the mathematics, are arbitrary. But since no one can visualize higher-dimensional knots anyway—the *only* representations of them are computationally reduced equations—it is an article of faith, not evidence, that the laws which describe them, independently of how we reduce them, are as hard as those of, say, Pythagoras's theorem (for which we do have adequate visualization models apart from the theorem). Paradoxes and anomalies do turn up, from time to time.

As Gardner is well aware, mathematics is not static; new discoveries are made, new theorems are added to our stock of old ones, whole new branches of mathematics are grafted onto the tree. The discovery of irrational numbers, for example, *"pushed along the social process of enlarging the way mathematicians decided to use the word 'number'"* [*ibid*., "How Not To Talk About Mathematics", p.287]. "Deciding to use" something or not to use it (non-Euclidean geometry, for example) may constitute a paradigm shift no less profound for mathematicians than heliocentricity proved to be for astronomy and theology. Every paradox and anomaly heralds a potential paradigm shift. One reason why mathematics describes the world so well is that, whenever it didn't, we fine-tuned it until it did. In some cases, mathematics is more like a suit of clothes which we modify to fit our own designs, than it is like a fossil which we find complete and unalterable.

As Gardner is also well aware, pure mathematics is much neater than applied mathematics. Consider the concept of tolerance in surveying, for example. No matter how accurately a surveyor surveys a field, there is always a margin of error, or tolerance. On the map, the zoning authority draws a rectangle, but in the field the surveyor finds that it has a short side. That is just the way the world is. Here, we should not say that mathematics describes the world well, but rather too well. It describes an idealized world, one that does not in fact exist.

Let us now reconsider Gardner's question, whether the moon would still be spherical if no humans existed. Are we all agreed that it *is* spherical? If we are doing pure mathematics, then, no, the moon isn't spherical—it is a dusty rock in space. If we are doing pure mathematics, then the one and only thing which is spherical is: a three dimensional surface of all points equidistant from a fixed point. That is, an abstraction. A definition. Not an object in the real world capable of being measured or surveyed. What if we are not doing pure mathematics but applied mathematics (e.g., cartography)? Is the moon a sphere? No. It is an irregular solid. It only approximates to being spherical, and *we* determine how close anything must come to being spherical in order for us to decide to call it so. We decided, at a certain point in history, to stop calling the earth a sphere and to start calling it a pear (we made the decision on the basis of measurements taken from satellites in orbit). And what shape is a pear? The definition of "pear-shaped" is rather more loose than that of "sphere". It is more like a suit which we cut to fit us, than it is like Pythagoras's theorem.

What shape is an asteroid? Show me the asteroid and I'll tell you what shape it is. What shape is a potato? Show me the potato and I'll tell you what shape it is.

It should be evident that the proviso "if no humans existed" has no bearing on the issue. Either the moon has whatever shape it has independently of how its shape is classified, or it has not. That is the issue between realists and non-realists. The realist says "yes" unequivocally, and thereby commits himself to a metaphysically fixed set of classes (including classes of shapes). The non-realist says "well, maybe; it depends..." Depends on what? If you say "mind" you are a subjective idealist; if you say "minds" you are a collective idealist; if you say "culture" you are a cultural relativist, and so on. Clearly, the same considerations apply to a nebula or a nautilus shell as to planetary bodies.

Mathematical realism is trivial if all it tells us is that something has whatever shape it has, independently of how it is classified. Ultimately, every object in the universe is unique, and so, therefore, is the shape of every object in the universe. But we want to know *what* shape something has, and because it is generally impractical to treat every case as unique and unrepeatable, we employ systems of classification. And so the human element necessarily enters the equation. By what criteria do we decide to classify a newly discovered asteroid among the spheroids, or the potatoids, or the higher-dimensional knotoids? By what criteria do we decide to classify a new theorem as a paradox, as an exception to the rules under limiting circumstances, as a new rule, or as a major paradigm shift? Neither the nautilus nor the nebula nor the theorem whispers to us what *sort* of thing it is or to what class it belongs. It may indeed belong to many different classes, and which one is the right one is a question not of metaphysics, but of aptness to human purposes. I am quite happy to grant that an object has whatever shape it has independently of how we classify it; but nothing whatever follows from this.

If a mathematical realist balks at the idea of having to swallow a metaphysically fixed set of classes (including classes of shapes), then I think he will find that he is lost somewhere on Gardner's neo-Laffer curve [*ibid*., "The Laffer Curve", see Gardner's illustration p.133]. Two limiting cases can be identified, namely, the two points of zero return on investment; but everything in between is a hopeless snarl where you can neither plot your actual position nor navigate with certainty to a more advantageous one. The alternative, of course, is to modify your realist stance and admit that classes (including classes of shapes) are not absolutes (metaphysically or otherwise); they depend on the mind, or minds, or culture, or language.

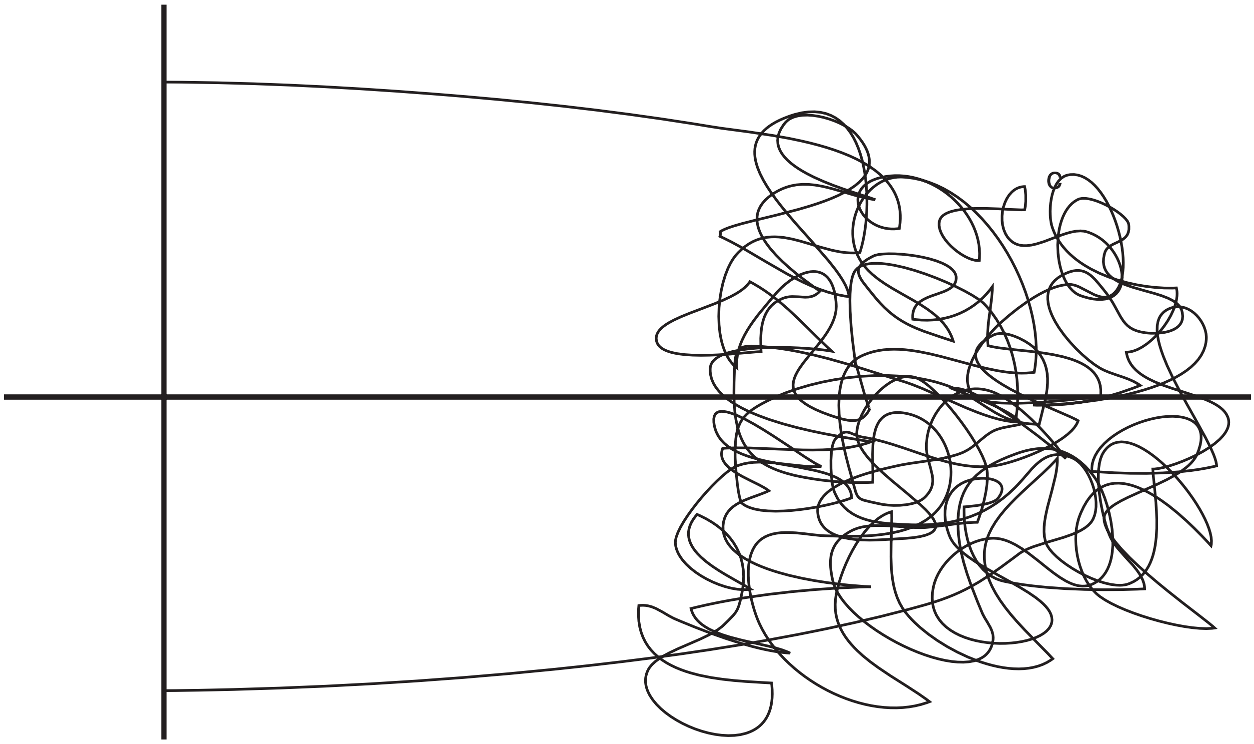

Fig.2 Gardner's Neo-Laffer Curve

If you want to assert that mathematics works because we got it from nature, because we learned it by observing natural patterns and generalizing them, then you have to wonder whether the world, like mathematics, is a formal system with axioms. (Spinoza, for example, certainly thought so.) The alternative, which Gardner dismisses as untenable, is that mathematical concepts are wholly mental constructs which have no objective reality or existence independent of minds. Let us consider the fact that the Greek and Roman number systems lacked the zero. They did not merely lack a numeral (a symbol, a doodle in their notation), but the concept of a place holder. If mathematical concepts such as sphericity, spirality, pairedness, and handedness have objective reality and exist independently of minds, as Gardner maintains, then I should like to know where we got the concept of a place holder. Does the world have place holders? Certainly there are bigger things and smaller things in the world. But are there orders of magnitude in the world, and does the world express them by means of place holders? Is that where we got them, and is that why they are so very useful? Gardner claims to be a

Platonist—which is an odd thing for a realist to say, because Plato is the archetypal idealist—, and a thorough-going Platonist must answer “yes” here. He must assume that the world has place holders (particular instantiations of that universal concept or law). Where are they, please? Can he show us one? Can he show us a gap in nature which not is merely a gap, but *representative of an order of magnitude?*

Can he show a natural instantiation of the concept “the square root of minus one”?

A thorough-going Platonist will have to maintain that not only sphericity and pear-shapedness have a separate and independent existence, but every shape whatsoever, however irregular and lopsided, including, for example, the shape of my nose both before and after I had a wart burned off it. This is to create a superfluous duplicate reality, full of shapes but devoid of content. The supposition of a nether realm full of all possible shapes (but devoid of substance) is just as fantastic as the supposition some cosmologists make, and for which Gardner chides them, that duplicate universes or alternate time-lines are created at every quantum event.

If, on the other hand, he maintains that the concept of place holders pertains only to our notation and has no objective reality independent of our notation, then I would like to see a principle of discrimination between those mathematical concepts which have objective reality and those which do not. Will it turn out that the definition of sphericity is objective but that the definition of pear-shapedness or lopsidedness or the shape of my nose is not? This would be arbitrary; as arbitrary as he claims collective solipsism to be.

Another question of general philosophical interest is how we are to derive mathematical laws from empirical laws at all, or apply analytical truths to empirical observations. Nature is full of patterns, and that, Gardner asserts, is why mathematics describes nature so well. By “pattern” he means something very like mathematics. But mathematical patterns are fundamentally different from natural ones. Empirical laws are generalizations based on observed causes and effects. Mathematical laws, however, are not generalized observations of how people add and subtract. They consist in rules and axioms in a formal system; in principle, they are translation algorithms for generating symbols from other symbols. It is not self-evident how you get the one kind of pattern from the other, or what it is for the one to be “based on” the other. What makes mathematical laws so convincing is that they are analytic—they are *defined* as valid. But that is exactly what empirical laws are not. How can something which is analytic describe something which is contingent? The solipsist is surely wrong to collapse the two categories into one and to assert that there is in principle no difference (or only a difference of degree), but I fault the realist for his naive assumption that the relation between them is perfectly obvious and in no need of clarification. To quote Gardner again: *“Any two cultures, isolated from each other, that develop a system for measuring the two sides of a right triangle and calculating the hypotenuse will discover the same Pythagorean rule, because that is how the world is structured.”* [*ibid*., “Mathematics & The Folkways”, p.268]. I protest—it is an article of faith that that is how the world is structured. I counter: that is how *calculation* is structured! Of course the two cultures will get the same result—I do not for an instant doubt *that.*

In cryptology, there are algorithms which transform strings of characters into other strings and back again. For example, “the quick brown fox jumps over the lazy dog” might be transformed into “907&%$£fvbNHG_=+/?>,#@[hgV”. This might, for example, be a password transmitted over a network and unencrypted at the other end. If the same algorithm is used to encrypt the same seed a second time, quite a different string may result, but so long as the original seed is recovered at the other end, the algorithm is useful. Now imagine someone saying “this encryption is eerily exact; we always get the same result, backwards or forwards” and then concluding from that, “therefore there must really be a quick brown fox which jumps over a lazy dog." There you have Gardner's position in a nutshell. The point is that we have well-defined uses for such algorithms, and if they were not eerily precise—if the original string did not reliably come out the other end—then they would be useless to us. They would not be encryption algorithms but something else (pseudo-random string generators, or something). Now imagine that someone had discovered such an algorithm, one which would reliably generate one string from another and back again, but without having any use for it; not, that is, an encryption algorithm for encoding plain text over a network, but merely as an idle transformation of "&8*(+-" into "lkj^%$£" and back again. And now imagine someone claiming that "&8*(+-" must really exist as an independent object in the world, whether or not there is any algorithm to generate it. There you have Gardner in a nutshell.

Gardner himself is aware of the problem, for, as he says, the theorems of Euclidean geometry are analytically true, not because they fit the world, but because they are part of a formal system. And *"a formal system ... says nothing about the world 'out there'. ... Euclidean geometry says nothing at all about the structure of physical space. ...Insofar as geometry applies to the outside world, it loses its certainty. By the same token, it is necessarily true only when its empirical meanings are abandoned."* [*ibid*., "How Not To talk About Mathematics", pp.285, 291]. How then do we know that we are applying the right concept of (formal, Euclidean) space to physical space, and how do we know we are applying it in the right way? Gardner refers to *"what Carnap called correspondence rules which link such ideal concepts as points and lines to observed physical structures."* But this is a fudge. First, because it begs the question to assert that we know which formal abstractions correspond to physical structures: points and lines are too easy—it gets tough when we search about for physical structures corresponding to place holders, the square root of minus one, and higher-dimensional knots, for example. It gets tough when we have to decide how closely something has to approximate to being spherical, or pear-shaped, before we stop calling it "irregular" or "lopsided". If you are navigating, for example, Euclidean space is the *wrong* space; not because its laws are invalid, but because the world is not flat. And second, the little word "link" slips in unnoticed. It is *not* self-evident what it is to *link* a physical point with a mathematical point—if you think it is, then you are bounding headlong into Zeno's paradoxes (you can't get halfway because, before you do, you have to get of the half, and so on to infinity). There is such a thing as making the wrong link between physical and mathematical points, or applying the link in the wrong way, and Zeno's paradoxes are an example of what happens when you do.

I do not for an instant doubt the validity of the commutativity of addition and multiplication. What I doubt is, 1) that these are corroborated or validated or instantiated by objects or processes in nature, and 2) that it is self-evident when and how to apply such concepts to objects and processes in nature. One of the most obvious characteristics of mathematical operations is their reversibility; numbers can be multiplied and divided without diminution. One of the most obvious characteristics of natural processes is their irreversibility; you do not get clay again by putting a crock back into the kiln.

What Carnap called correspondence rules are either another formal system, which has as little to do with the real world as geometry has, or they are empirical statements based on observations and therefore not certain. In either case, we are still left wondering what the link is.

There is, for example, a link between one degree Celsius and one gram of water—a unit of energy has been defined as the amount of energy required to raise that amount of water that number of degrees Celsius. But once the water reaches 100 degrees, the same amount of energy does not raise it one degree more. At that point the link is broken, and it requires orders of magnitude more energy to convert the water to steam. How much more is a matter of empirical observation, not ‘correspondence rules’. Evidently there will have to be as many links as there are applications (or possible applications) of mathematics to the real world, and we are back on Gardner’s neo-Laffer curve.

As Gardner himself points out, addition in nature is not always so clear cut as it is in arithmetic. If you add a drop of water to another one, you do not get two drops—you get one big drop. And if you pour a liter of alcohol into a liter of water, you do not get 2 fluid liters—you get less. He adds the proviso that you have to keep regarding the units *as units* in order apply the arithmetical sense of addition to rain drops (or to quantities of fluids which dissolve in other fluids). But just as nature does not whisper to us what *sorts* of things things are, neither does it whisper to us when to regard something as a unit rather than as a type (sort or class). That is why we ought to be skeptical of the realist assumption that mathematical laws are simply derived from observation of natural laws. It is anything but simple. You already have to have reached a decision (perhaps tacitly) to regard the phenonema under observation *as types* or *as units* in order to be able to derive anything from them, and that means you are interpreting an assumption or human purpose into the world in order to derive a mathematical law from it. It is a curious dance requiring both objective discovery and mental projection.

I can think of a perfectly ordinary case in which 2 + 2 does not equal 4. There is a children’s riddle which goes like this: two mothers and two daughters are having lunch. How many people are having lunch? The answer is three. That is the sort of case where correspondence rules come off the rails. Of course, one of the terms in the equation is doing double-duty, but that in no way vitiates the point. Sometimes things do do double-duty, and you have to know this to correctly apply an arithmetical function. Knowing when this is the case is not itself an arithmetical function, and this observation is sufficent to torpedo any putative a priori correspondence rule.

I can think of a perfectly ordinary case in which 3x12 does not equal 36. If one egg costs 3 cents, then 12 eggs will cost 36 cents; but if it takes 3 minues to boil an egg, how long does it take to boil a dozen eggs? Three minutes!

Addition is commutative. But not always. If you are navigating, you add and subtract various factors such as current direction and speed, wind direction and speed, magnetic deviation from true north depending on where you are on the globe, magnetic deflection of the compass due to the amount of metal on the boat, and so on. If you are doing pure mathematics, it does not matter in what order you add or subtract figures; but if you are navigating it does. When navigating, these factors are to be added and subtracted from each other in a quite specific order; if you don’t, you end up somewhere else. Moreover, the reason why addition is not commutative in navigation cannot be explained by appeal to any mathematical principles or laws. It can only be ‘explained’ by saying that while addition is commutative, the world is not.

“Mathematics fits the world with eerie exactitude,” Gardner writes. But let us not be overawed by this: mathematics *is* eerily exact, whether it fits anything or not, for that is the nature of formal systems (it is the same with chess and encryption algorithms). The world can be exactly described only insofar as it is determinate, but it is an article of faith, not evidence, that it *is* determinate; if it should turn out that some aspects of it are not determinate, then we must be prepared to accept the possibility that mathematics will not be an appropriate method of description for such aspects. Exactitude by itself is no guarantor of truth, and as Gardner himself points out with devastating accuracy, the pseudo-exactitude of some branches of mathematics—especially economics—is not merely ridiculous, but dangerous if enacted as public policy. In the physical sciences, too, when mathematics fits too perfectly, cautious skepticism is advised; when the numbers fit too well, it may be an indication that we have constructed an elaborate tautology.

Sometimes it is mere coincidence when patterns in mathematics match patterns in nature. There was a time in the history of astronomy when the number of known planets in the solar system equaled the number of Platonic solids. This appeared to prove an elegant theory: the universe was a 'harmony of spheres', there were just so many Platonic solids, therefore, there could only be just so many planets. The theory was elegant, the mathematics worked out perfectly—and the whole thing was bosh. The lesson to be learned is that when you have an elegant theory and all the numbers fit, the important phase of scientific investigation is just beginning, not just concluded; we still have to determine whether the pattern we think we have discovered is in fact the cause of the phenomenon, or one of its effects, or merely a coincidence.

Let us briefly consider another area of mathematics, one that features in many of Gardner's other writings: quantum mechanics. As Gardner himself is fond of saying, things at the quantum level are just plain weird. Gardner accepts two propositions as now generally accepted by most physicists: 1) that quantum events are undetermined until someone measures them (he cites the EPR paradox: two photons are emitted in opposite directions; they must have opposite spins, but which has which appears to be undetermined until one or the other is measured); and 2) that matter has vanished on the quantum level, "leaving only patterns" [ibid., "How Not To Talk About Mathematics", p.281]. Quantum weirdness *"springs from the fact that the waves of QM are mathematical fictions, abstract waves of probability in multidimensional spaces constructed solely to describe quantum systems."* [ibid., "Quantum Weirdness", p.29]. Nonetheless, Gardner is convinced that the universe exists independently of minds, that it has patterns which exist independently of minds, and that mathematics correctly describes these patterns (or some of them anyway)—that, for him, is the essence of mathematical realism. But if matter has vanished at the quantum level, to be replaced by measurements and probabilities and what he himself calls "mathematical fictions", then he has tacitly abandoned the fundamental tenet of mathematical realism, namely that mathematics describes an independently existent world. He has tacitly accepted the 'solipsist' position, at least at the quantum level, that mathematics *is* the world it describes. Mathematics describes the world in the sense in which a compass describes a circle: namely, by tracing round its structure, just as you run your finger round the rim of a glass. But, at the quantum level, there is no longer any finger or glass, there is just the metaphorical motion which the mathematical law itself defines to be a circle.

Mathematical realism holds if there are patterns in nature which are correctly described by mathematical functions; it collapses into solipsism when the patterns in nature vanish and are replaced by sheer speculation. An indeterminate pattern is no pattern at all, and to say that a natural pattern is undetermined until it is measured, so far as the evidence goes, is the same as saying that it is indeterminate, and that is to collapse it into collective solipsism.

To a cautious skeptic, it appears to be merely an article of faith, not evidence, that the mathematical metaphors employed at the quantum level are really mapping real patterns which exist independently of anyone's trying to map them. Is it not possible that they are emergent patterns? At the quantum level, it is not at all evident that we are learning mathematics from the world, generalizing particular cases to arrive at laws which have an objective, universal validity; on the contrary, it looks as though we are imposing a pattern on a vague fog of expectations, simply because we want to see a pattern. It looks like a replay of the Platonic-solids theory of the solar system: any pattern will do, so long as somebody manages to bash the equations into a wonderful symmetry. That is tantamount to abandoning the correspondence theory of truth and adopting a coherence theory.

As cosmologists plunge into ever smaller, ever more abstract realms, the realms of superstrings and interior spaces and multiple universes branching off into other dimensions at every quantum event, it cannot be verified whether the mathematical patterns employed actually describe patterns occurring in nature independently of minds. The mathematical patterns may indeed be stupendous and gorgeous, but it is no longer science. Gardner seems to agree and quotes with approval: *"Howard Georgi calls string theory 'recreational mathematical theology'... 'not a theory in the usual sense,' declared Julian Schwinger, 'but an aesthetic and emotional glow about how things would work if only we could compute them.'"* [ibid., "Superstrings", p.86]. I call it intellectual autoeroticism. The question is, where do you draw the line between quantum weirdness and sheer fantasy?

No two flowers, snowflakes, or sunsets are identical. Each thing is like a new-born babe: unique and unrepeatable. To comprehend things we must generalize them, and that means we selectively ignore their unique features. We impose classifications and notations. Gardner, and empirical realists generally, claim that laws hold sway throughout the buzzing, booming confusion of unique instances, regardless of what the classifications and notations are, and that the business of science is to discover and formulate those laws. So far so good. Recognizing patterns in nature is a good start. But when we see a pattern, we are strongly tempted to suppose that we are seeing the force which generates it, though this is not necessarily so. Some patterns are fabricated by our own minds, by our language, or by our culture. Some natural patterns are just there—they don't *do* anything. Patterns tip us off to the likelihood of empirical laws, but are no guarantee of them. The hard work is to distinguish the ones which are in fact operational in nature from the ones which captivate us merely because they make a nifty system of notation or classification. "Realism" is too easy—too naive.

Gardner quotes Bohr (more than once) that a great truth is a statement whose opposite is also a great truth. I draw a corollary: if the negation of a statement is incoherent, then the assertion of it is trivial. Mathematical realism holds that mathematics describes the world so well because that is the way the world really is. But try negating the assertion. Who would propound a theory, on the one hand, and then say, on the other, 'but the world isn't really like that'? If he knew the world wasn't really like that, he wouldn't have propounded the theory.

I do not for half an instant believe that the cycles of the seasons or the commutativity of addition are figments of our imagination or that we could change them by fiat. Anyone who thinks that these things can be changed simply by human say-so has my hearty welcome to plant his seed at the wrong time and see what there is to eat when it comes time to harvest. But to claim that the laws of mathematics are the way they are because that is the way the world is, is to say nothing very significant. It is more or less the same as saying that language describes the world because that is the way the world really is. But what if it weren't? Would language make no sense at all? We cannot make sense of the possibility, nor is there any conceivable method for discovering, that the world is wholly and radically different from what we think it to be (either linguistically or mathematically). And that means first, that realism is unfalsifiable, and second, that it is trivial (since its negation is incoherent).

The realist *thinks* the world exists whether or not he thinks so—that is what realism comes down to. I am prepared to accept that as a declaration of faith, but as a foundation for a philosophy of mathematics, it is too thin ice to be skating on. If collective solipsism [idealism] asserts that the world exists *because* we think so, then I am prepared, with Gardner, to dismiss it as preposterous and monumentally egotistical.

Gardner apparently has a good deal of respect for Wm. James's pragmatism; I have, too, but I think he gives James short shrift on a particular point:

There are, James writes, coats and shoes that 'fit' backs and feet even though they are not yet made. "In the same way countless opinions 'fit' realities, and countless truths are valid, though no thinker ever thinks them." ... this includes countless scientific facts and laws not yet discovered. To the anti-pragmatist ... these [not-yet discovered truths] are the fundamental ones. To the pragmatist, they are "static, impotent, and relatively spectral" until they are verified by human experience. For a Jamesian, the thousandth decimal of pi was a ghost that did not spring into full-blooded reality until someone calculated it. "To attribute a superior degree of glory to [an unverified truth] seems little more than a piece of perverse abstraction worship." What James seems to be claiming here is that although facts about the world, and even theorems of mathematics, exist in some vague way before they are discovered, as soon as they enter human experience they acquire a stronger reality. I can think of few philosophical tasks less rewarding than defending the view that the planet Neptune became more real after humanity knew it existed, or that a giant prime becomes more real when it is proved to be prime. [ibid., "How Not To Talk About Mathematics", Postscript, p.293]

Of course the planet Neptune did not become more real as a result of its having been discovered by humans. James's point is not that unknown facts become more real (or unknown truths more true) as soon as we know them. James's claim is not that the metaphysical claim that facts are changed (or brought into existence) by being known. Where we have reason to suspect that a hypothesis may be true and we have means of verifying or falsifying it—for example, when we have noted perturbations in the orbits of other planets which can be plausibly accounted for by supposing the existence of another planet, and we can calculate roughly where the supposed planet would have to be to cause such perturbations—, *then* it makes sense to talk about an undiscovered truth which lies in wait for us, which 'sleeps'. That is the point of pragmatism. But to talk about unknown truths in general as having an independent existence is just as pointless as saying that all winning lottery tickets exist even if no one ever prints them.

postscript

Mr. Gardner defines some of his terms, especially "realism", idiosyncratically; in my article above, I made an effort to take his terms in the sense in which he uses them. Here I should like to put the record straight. "Realism", as it is used in the history of philosophy, is the position that the world is what it appears to be. That is, it consists of ordinary material objects evident to the senses: rocks, trees, rivers, mountains, people, "the furniture of the universe" as Russell put it. The question whether the world exists independently of minds (or the mind of God) does not arise in realism; to a realist, the question is silly and he is not bound to answer it.

Idealism is opposed to realism. Idealism is the position that the world consists of ideas or concepts, things not evident to the senses and radically different from material objects. What we perceive to be objects are shadows or illusions or epiphenonema.

Idealists divide themselves into two camps depending on how they answer the crucial question, which they are bound to answer, whether ideas exist independently of minds (or the mind of God). Those who answer that ideas exist independently of minds are Platonists. A Platonist is not a realist, and neither was Plato—for him, matter had only a shadowy, second-rate existence compared to Ideas. He maintained that the ultimate constituents of reality were radically different from that which appears to the senses—that is what makes him an idealist and not a realist—and his further assertion that these ultimate constituents exist independently of minds is what makes him an idealist of a particular sort, namely, a Platonist.

Those who deny that ideas (the ultimate constituents of reality) exist independently of minds (or the mind of God) fall in with George Berkeley; for him, matter had no existence at all. Gardner calls people who maintain this position “collective solipsists”.

In the philosophy of mathematics, these terms are also carefully defined and distinguished: *“mathematical realism is the view that the truths of mathematics are objective, which is to say that they are true independently of any human activities, beliefs or capacities. Mathematics is the study of a body of necessary and unchanging facts, which it is the mathematician’s task to discover, not create.”* [Concise Routledge Encyclopedia of Philosophy, entry ‘Realism in the Philosophy of Mathematics’] An important form of mathematical realism is Mathematical Platonism, which is the view that mathematics is about a collection of independently existing objects. In the philosophy of mathematics, the opposing position, “anti-realism”, is also clearly defined; it does not mean what colloquial usage might suggest, some sort of incoherent mental derangement. It means something both sane and clearly formulable: namely, that only what can be proven is true (in *mathematics*, whatever one may think about *empirical* truths). Mathematical realists claim that truths are true independently of proofs of them, just as empirical facts exist independently of being known; *anti*-realists claim that the truth of mathematical truths *consists* in their provability (for that is what it is to be a formal system, quite different to empirical truths). The anti-realist camp objects that what the realists claim to be facts and objects *“are inaccessible to us and bear no clear relation to the procedures we have for determining the truth of mathematical statements.”* [Concise Routledge Encyclopedia of Philosophy, entry ‘Realism in the Philosophy of Mathematics’]

My own view is that it is highly misleading to speak of mathematical facts and mathematical objects as if they were white swans which correlate to and thereby render the statement ‘swans are white’ true. If there is any object which correlates to a sphere, it is a collection of points equidistant to a point in three-dimensional space, namely a definition—which is an object in the grammatical sense. That is, it is an object of discourse in and for mathematics. To hypostatize it is a crude blunder. What makes it seem to be an object in the world, and what often leads us to speak as if it were, is that we have a use for it in the world, as we have a use for encryption algorithms. The quality of eerie precision is what we require of it; without that, it would not be useful (not for that purpose anyway). In the case of an equation describing a higher-dimensional knot, for example, there is no ‘object’ which correlates to it at all; there is only the equation. There is only the cypher procedure for transforming one symbol-string into another one. If someone insists on populating the universe, or ‘all possible worlds’, with all possible symbol-strings, whether or not they are ever generated by algorithms, then it is misleading for him to call this position realism.

I shall now quote a passage of Mr. Gardner’s which unequivocally shows into which camp he falls: *“...the human mind is made of molecules, which are in turn made of atoms, which are in turn made of electrons, protons, and neutrons. The protons and neutrons are made of quarks. What are quarks and electrons made of? Nothing except equations. Let’s face it. You and I, at the lowest known level of our material bodies, are made of mathematics, pure mathematics, mathematics uncontaminated by anything else.”* [*ibid.,* “Computers Near The Threshold?”, p.446] The gist of it is this: matter does not consist of matter all the way down; at the bottom, it consists of something else. It consists of number (an idea first proposed by Pythagoras). It is not evident that the human mind consists of equations. It is not evident that matter consists of equations. It is not evident that the mind is the same as the brain, and that both consist of equations. To maintain this is idealism, pure and simple. Gardner is wrong when he claims that this is realism; it is not. He bases his claim to the word “realism” on his contention that numbers and mathematical laws exist independently of minds. But this, as I have pointed out, is not an issue for realists—it is not *this* which distinguishes realists from nonrealists. If Gardner claims that matter and minds consist of mathematics, and that mathematics exists independently of minds, then what he is is an idealist of the Platonic sort. And this is just what he does say in other places.

Mr. Gardner values clarity and precision of formulation. So do I. When he redefines “realism” to mean “Platonism”, he subverts those aims. Reducing the mind to an equation independent of minds is not realism, it is irrealism.

Gardner poses the question whether the moon would still be spherical even if no minds existed. Let us pose the question one step harder: would sphericity still exist even if no matter whatsoever existed? Neither moons nor minds/brains? Would the shape of my nose exist if no noses ever existed? If he says no, then he falls into the “collective solipsist” camp. If he says yes, then he commits himself to the chimerical existence of a superfluous duplicate reality filled with empty notions and devoid of content. This is irrealism, pure and simple.

Mr. Gardner’s final argument in favor of realism is this: *“[Hardy’s] realism is certainly shared by most professional mathematicians, including those working on the foundations of mathematics. Just to be sure I was not biased in this opinion, I phoned my friend Raymond Smullyan, an expert on formal systems who also happens to be a Taoist. My first question was ‘Do you consider yourself a realist?’ He replied, ‘Of course.’ My next question was ‘Among today’s leading authorities on set theory who are doing creative work in the field, how many would you say are anti-realists?’ Smullyan said: ‘Almost none.’”* [*ibid.,* “How Not To Talk About Mathematics”, postscript, p.292]

Unfortunately, it is not clear whether he and his colleague were taking “realist” and “anti-realist” in the strict sense as defined by professional philosophers of mathematics (that truth consists in provability), or rather more colloquially (as a silly notion no sane, clear-thinking person would publically admit to entertaining). In either case, if the number of people doing creative work in a given field is to be the criterion of validity for work in that field, then we must conclude that Gardner ultimately maintains a consensus theory of truth. If “most professional mathematicians” are to constitute the criterion of truth, then we had better poll a statisically significant sample of them, not just Mr. Gardner’s friend.

It is well to remember that consensus changes and that it is sometimes goaded out of complacent inertia by ideas which, at first, seem wildly counter-intuitive and anti-logical; such changes were wrought by the work of Cantor, Goedel, and Einstein, for example.

Consensus theorist, realist, irrealist, Platonist, collective solipsist? Will the real Mr. Gardner please stand up? Perhaps quantum indeterminacy permeated his life more completely than anyone suspected. Perhaps there is no real Martin Gardner; perhaps he is an imaginary number in the mind of a dream figure in an imaginary novel in one of Borges’s infamous, infinite libraries of Tlön. ‘Upward, beyond the moondling, it gardnered...’

end of file